

Torus fixed points of moduli spaces of stable bundles of rank three

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1 Introduction

The aim of this paper is to extend the main result of [11], which determines the Euler characteristic of moduli spaces of stable bundles of rank two on the projective plane, to the case of bundles of rank three. Therefore, we first discuss some results of [11], [12] and [13] and some general results about vector bundles and their moduli spaces. Klyachko proved that toric bundles of rank n correspond to filtrations of an n -dimensional vector space. These filtrations can also be understood as representations of the subspace quiver. Since the stability condition can be transferred, the moduli spaces of stable representations can be identified with some fixed point components of the moduli space of bundles. Toric bundles of rank three correspond to filtrations of a three dimensional vector space.

The length of the arms of the subspace quiver can be used to determine the Chern classes of the corresponding torus fixed points. Investigating the moduli spaces of (semi-)stable representations it turns out that the second Chern class varies depending on how many two-dimensional subspaces the one-dimensional ones contain. For the discriminant of bundles of rank three on the projective plane we either have 0 or 4 mod 6. In the first case we first analyse the polystable points which are representations which can be decomposed into representations of the same slope. It turns out that the existence and number of those points just depend on the length of the arms of the subspace quiver. We also investigate those stable points of the moduli space having different second Chern class. In the second case there exist only stable points, i.e. there are no polystable points, so we just have to consider the latter cases. In both cases it turns out that the moduli spaces of semistable representations are projective lines and that there exist only finitely many of the described points so that it is easy to calculate the Euler characteristic of the fixed point components.

The stability condition reduces to a system of linear inequalities which can be solved using standard methods from e.g. [20]. After stating how to determine the solutions of such a system we will see that in our case they

correspond to solutions of some quadratic equations so that the generating functions of the Euler characteristic of the moduli spaces of stable bundles of rank three on the projective plane can be determined.

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2 Notation and terminology

2.1 Representations of quivers

Let k be an algebraically closed field.

Definition 2.1 *A quiver Q consists of a set of vertices Q_0 and a set of arrows Q_1 denoted by $\alpha : i \rightarrow j$ for $i, j \in Q_0$. In this situation i is called the tail and j the head of the arrow α . A quiver is finite if Q_0 and Q_1 are finite.*

An oriented cycle in Q is a set of arrows $\alpha_1 : i_1 \rightarrow i_2, \alpha_2 : i_2 \rightarrow i_3 \dots, \alpha_n : i_n \rightarrow i_{n+1}$ such that $i_1 = i_{n+1}$.

In the whole paper we only consider quivers without oriented cycles. Define the abelian group

$$\mathbb{Z}Q_0 = \bigoplus_{i \in Q_0} \mathbb{Z}i$$

and its monoid of dimension vectors $\mathbb{N}Q_0$. We introduce a non-symmetric bilinear form called the Euler form on $\mathbb{Z}Q_0$. Define

$$\langle d, e \rangle := \sum_{i \in Q_0} d_i e_i - \sum_{\alpha : i \rightarrow j} d_i e_j.$$

A finite-dimensional k -representation of Q is given by a tuple

$$X = ((X_i)_{i \in Q_0}, (X_\alpha)_{\alpha \in Q_1} : X_i \rightarrow X_j)$$

of finite-dimensional k -vector spaces and k -linear maps between them. The dimension vector $\underline{\dim} X \in \mathbb{N}Q_0$ of X is defined by

$$\underline{\dim} X = \sum_{i \in Q_0} \dim_k X_i i.$$

Let $d \in \mathbb{N}Q_0$ be a dimension vector. The variety $R_d(Q)$ of k -representations of Q with dimension vector d is defined as the affine k -space

$$R_d(Q) = \bigoplus_{\alpha : i \rightarrow j} \text{Hom}_k(k^{d_i}, k^{d_j}).$$

The algebraic group

$$G_d = \prod_{i \in Q_0} Gl_{d_i}(k)$$

acts on $R_d(Q)$ via simultaneous base change, i.e.

$$(g_i)_{i \in Q_0} * (X_\alpha)_{\alpha \in Q_1} = (g_j X_\alpha g_i^{-1})_{\alpha: i \rightarrow j}.$$

The orbits are in bijection with the isomorphism classes of k -representations of Q with dimension vector d .

In the space of \mathbb{Z} -linear functions $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}Q_0, \mathbb{Z})$ we consider the basis given by the elements i^* for $i \in Q_0$, i.e. $i^*(j) = \delta_{i,j}$ for $j \in Q_0$. Define

$$\dim := \sum_{i \in Q_0} i^*.$$

After choosing $\Theta \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}Q_0, \mathbb{Z})$, we define the slope function $\mu : \mathbb{N}Q_0 \rightarrow \mathbb{Q}$ via

$$\mu(d) = \frac{\Theta(d)}{\dim(d)}.$$

The slope $\mu(\dim X)$ of a representation X of Q is abbreviated to $\mu(X)$.

Definition 2.2 *A representation X of Q is semistable (resp. stable) if for all proper subrepresentations $0 \neq U \subsetneq X$ the following holds:*

$$\mu(U) \leq \mu(X) \text{ (resp. } \mu(U) < \mu(X)).$$

Denote by $R_d^{ss}(Q)$ the set of semistable points and by $R_d^s(Q)$ the set of stable points in $R_d(Q)$. In this situation we have the following theorem based on Mumford's GIT and proved by King, see [18] and [10]:

Theorem 2.3 *We have:*

1. *The set of stable points $R_d^s(Q)$ is an open subset of the set of semistable points $R_d^{ss}(Q)$, which is again an open subset of $R_d(Q)$.*
2. *There exists a categorical quotient $M_d^{ss}(Q) := R_d^{ss}(Q)/G_d$. Moreover, $M_d^{ss}(Q)$ is a projective variety.*
3. *There exists a geometric quotient $M_d^s(Q) := R_d^s(Q)/G_d$, which is an open smooth subvariety of $M_d^{ss}(Q)$.*

Remark 2.4

- For a stable representation X its orbit in $R_d(Q)$ is of maximal possible dimension, see [10]. Since the scalar matrices act trivially on $R_d(Q)$, the isotropy group is at least of dimension one. Therefore, if the moduli space is not empty, we get for its dimension

$$\dim M_d^s(Q) = 1 - \langle d, d \rangle.$$

2.2 Moduli spaces of vector bundles on the projective plane

In this section we treat basic results concerning the theory of vector bundles on projective spaces based on [19], [14] and [7].

Let $\mathcal{E} \neq 0$ be a vector bundle on \mathbb{P}^n . Denote by $c_i(\mathcal{E})$ its i -th Chern class and by $\text{rk}(\mathcal{E})$ its rank. The Chern polynomial of \mathcal{E} is given by

$$c(\mathcal{E}) = \sum_{i=0}^{\text{rk}(\mathcal{E})} c_i(\mathcal{E}) t^i.$$

Define by

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E})}{\text{rk}(\mathcal{E})}$$

the slope of the vector bundle.

Definition 2.5 *A vector bundle \mathcal{E} on \mathbb{P}^n is semistable if we have*

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E})$$

for all coherent subsheaves $0 \neq \mathcal{F} \subseteq \mathcal{E}$.

The bundle \mathcal{E} is stable if we have

$$\mu(\mathcal{F}) < \mu(\mathcal{E})$$

for all proper coherent subsheaves \mathcal{F} with $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$.

A stable bundle with respect to this stability condition is also often called stable in the sense of Mumford-Takemoto or simply μ -stable.

Definition 2.6 *The discriminant of a stable bundle of rank r on \mathbb{P}^2 with Chern classes c_1 and c_2 is defined by*

$$D = \frac{1}{2r^2}(2rc_2 - (r-1)c_1^2).$$

In this situation we have the following theorem, see [2]:

Theorem 2.7 *Let \mathcal{E} be a semistable bundle of rank $r > 0$ and discriminant D on the projective plane. Then we have $D \geq 0$.*

There are lots of articles dealing with moduli spaces of stable vector bundles (or sheaves resp.) on projective varieties, see for instance [24], [16] or [17]. In this paper we are only interested in moduli spaces of μ -stable bundles on the projective plane. In particular, we are not interested in moduli spaces of semi-stable bundles. We do not describe the construction in detail and only need the following property of moduli spaces of μ -stable vector bundles on the projective plane, again see for instance [24], [16] or [17]:

Theorem 2.8 *Fix the first two Chern classes c_1, c_2 and the rank r . Then there exists a quasi-projective variety $\mathcal{M}(r, c_1, c_2)$ which parametrizes the isomorphism classes of μ -stable vector bundles \mathcal{E} on the projective plane such that $c_1(\mathcal{E}) = c_1, c_2(\mathcal{E}) = c_2$ and $\text{rk}(\mathcal{E}) = r$.*

In the following denote by $\mathcal{O}(1)$ the hyperplane bundle on the projective plane and its dual line bundle by $\mathcal{O}(-1)$. Also define

$$\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$$

if $k \geq 0$ and

$$\mathcal{O}(k) = \mathcal{O}(-1)^{\otimes k}$$

if $k < 0$.

If we consider a bundle of rank r on \mathbb{P}^n , its Chern classes c_t with $t \in \mathbb{N}$ satisfy $c_t = 0$ for $t > \min(r, n)$. For details see for instance [14].

Let \mathcal{E} and \mathcal{F} be vector bundles of rank r and s respectively on the projective space \mathbb{P}^n and let

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + a_i t)$$

and

$$c(\mathcal{F}) = \prod_{i=1}^s (1 + b_i t)$$

respectively with $a_i, b_i \in \mathbb{Z}$ be their Chern polynomials. Define c_i with $i = 0, \dots, r+s$ by

$$\prod_{i,j} (1 + (a_i + b_j)t) = \sum_{i=0}^{r+s} c_i t^i.$$

For the Chern polynomial of the bundle $\mathcal{E} \otimes \mathcal{F}$ we have, see [8] or [7]:

$$c(\mathcal{E} \otimes \mathcal{F}) = \sum_{i=0}^{\min(n, \text{rk}(\mathcal{E} \otimes \mathcal{F}))} c_i t^i.$$

If

$$c(\mathcal{E}) = \sum_{i=0}^{\text{rk}(\mathcal{E})} c_i(\mathcal{E}) t^i,$$

for the Chern polynomial of the dual bundle \mathcal{E}^* we have

$$c(\mathcal{E}^*) = \sum_{i=0}^{\text{rk}(\mathcal{E})} c_i(\mathcal{E}) (-t)^i.$$

Thus we obtain the following important property for the moduli spaces of stable bundles on the projective plane:

Proposition 2.9 *Twisting with a line bundle $\mathcal{O}(k)$ induces an isomorphism of moduli spaces*

$$\mathcal{M}(r, c_1, c_2) \cong \mathcal{M}(r, c_1 + rk, c_2 + (r-1)kc_1 + k^2 \frac{r(r-1)}{2}).$$

If $r \leq 3$, the moduli spaces only depend on the discriminant.

Proof. Let \mathcal{E} be a stable bundle with Chern classes c_1, c_2 and rank r . The rank of \mathcal{E} does not change after tensoring by $\mathcal{O}(k)$. From

$$\begin{aligned} c(\mathcal{E} \otimes \mathcal{O}(k)) &= (1 + (k + \frac{c_1}{2} - \sqrt{\frac{c_1^2}{4} - c_2})t) \\ &\quad (1 + (k + \frac{c_1}{2} + \sqrt{\frac{c_1^2}{4} - c_2})t)(1 + kt)^{r-2} \mod t^3 \end{aligned}$$

it follows

$$c(\mathcal{E} \otimes \mathcal{O}(k)) = 1 + (rk + c_1)t + (\frac{r(r-1)}{2}k^2 + (r-1)kc_1 + c_2)t^2.$$

Now an easy calculation implies that the discriminants of \mathcal{E} and $\mathcal{E} \otimes \mathcal{O}(k)$ coincide. Furthermore, it is easy to see that (semi-)stable bundles remain (semi-)stable when tensoring with a line bundle. For $r = 1, 2$ this suffices to prove the second statement.

Thus let $r = 3$. Because of the first part we may assume that $0 \leq c_1 \leq 2$. If $c_1 = 2$ we may take the dual and tensor with $\mathcal{O}(1)$ afterwards in order to get $c_1 = 1$. Note that the discriminant does not change when taking the dual. Under the assumption $c_1 \in \{0, 1\}$ it is easy to see that the moduli spaces only depend on the discriminant.

□

Fixing the rank to be r , these moduli spaces will be denoted by $\mathcal{M}(r, D)$ or simply $\mathcal{M}(D)$ if $r \leq 3$. By considering a torus action on the moduli space we can calculate the Euler characteristic via reduction to torus fixed points. The $(n+1)$ -dimensional torus $T = (\mathbb{C}^*)^{n+1}$ acts on the projective space \mathbb{P}^n via multiplication, i.e.

$$t \cdot (x_0 : x_1 : \dots : x_n) = (t_0 x_0 : t_1 x_1 : \dots : t_n x_n)$$

for $t = (t_0, t_1, \dots, t_n) \in T$.

Definition 2.10 *A vector bundle $p : \mathcal{E} \rightarrow \mathbb{P}^n$ is toric if T acts linearly on the fibres such that for each $t \in T$ the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{t} & \mathcal{E} \\ \downarrow p & & \downarrow p \\ \mathbb{P}^n & \xrightarrow{t} & \mathbb{P}^n \end{array}$$

Let E be a vector space. Then a descending \mathbb{Z} -filtration is defined as a chain of subspaces $E(i) \subset E$, $i \in \mathbb{Z}$, such that $E(i) \subset E(i-1)$. Denote the set of all filtrations of a vector space E by $\mathcal{F}(E)$. Analogously, we can define families of filtrations of a space E , i.e. $E^\alpha \in \mathcal{F}(E)$ for $\alpha \in I$ and an index set I .

Let \mathcal{E} be a toric bundle on \mathbb{P}^2 . The $T = (\mathbb{C}^*)^3$ -action has an open orbit containing all points $p = (x_\alpha : x_\beta : x_\gamma) \in \mathbb{P}^2$ such that $x_\alpha, x_\beta, x_\gamma \neq 0$. Let $E := \mathcal{E}(p_0)$ be the fibre of an arbitrary point p_0 in this orbit. Since \mathcal{E} is toric, we have $te \in \mathcal{E}(tp_0)$ for all $e \in E$. Now choose a generic point p_α from the coordinate line

$$X_\alpha = \{(x_\alpha : x_\beta : x_\gamma) \in \mathbb{P}^2 \mid x_\alpha = 0\}.$$

Define

$$E^\alpha(i) := \{e \in E \mid \lim_{tp_0 \rightarrow p_\alpha} \left(\frac{t_\alpha}{t_\beta}\right)^{-i} (te) \text{ exists}\}$$

and define E^β, E^γ analogously. Obviously this definition is independent of the choices of p_0 and p_α . Instead of the chosen rational function, we may also consider every other function with a pole of order i in X_α .

By this procedure for every bundle we get a family of descending \mathbb{Z} -filtrations of E . Indeed we obviously have:

$$\dots \subset E^\alpha(i+1) \subset E^\alpha(i) \subset E^\alpha(i-1) \subset \dots$$

with the additional property $E^\alpha(i) = 0$ for $i \gg 0$ and $E^\alpha(i) = E$ for $i \ll 0$. From [12] we get the following theorem:

Theorem 2.11 *The category of toric bundles on the projective plane is equivalent to the category of vector spaces with a family of descending \mathbb{Z} -filtrations $E^\alpha \in \mathcal{F}(E)$ with $\alpha = 1, 2, 3$ such that*

$$E^\alpha(i) = 0 \text{ for } i \gg 0 \text{ and } E^\alpha(i) = E \text{ for } i \ll 0.$$

Two filtrations $E_1, E_2 \in \mathcal{F}(E)$ are isomorphic if there exists a $g \in Gl(E)$ such that $gE_2(i) = E_1(i)$ for all $i \in \mathbb{Z}$. Similarly, two families E_1^α, E_2^α with $\alpha \in I$ for some index set I are isomorphic if there exists a $g \in Gl(E)$ such that $gE_1^\alpha(i) = E_2^\alpha(i)$ for each $\alpha \in I$ and each $i \in \mathbb{Z}$.

Let \mathcal{E} be a toric bundle on \mathbb{P}^2 given as a filtration E^α , $\alpha = 1, 2, 3$. The first two Chern classes of this filtration and the corresponding bundle respectively are given as follows, see [12]:

$$c_1(\mathcal{E}) = \sum_{i \in \mathbb{Z}, \alpha} i \dim E^{[\alpha]}(i)$$

where $E^{[\alpha]}(i) = E^\alpha(i)/E^\alpha(i+1)$ and

$$c_2(\mathcal{E}) = \frac{c_1(\mathcal{E})^2}{2} - \frac{1}{2} \sum_{i \in \mathbb{Z}, \alpha} i^2 \dim E^{[\alpha]}(i) - \sum_{\alpha \neq \beta, (i,j) \in \mathbb{Z}^2} ij \dim E^{[\alpha, \beta]}(i, j)$$

where $E^{[\alpha, \beta]}(i, j) = E^\alpha(i) \cap E^\beta(j) / (E^\alpha(i+1) \cap E^\beta(j) + E^\alpha(i) \cap E^\beta(j+1))$. The twist by again a line bundle corresponds to a shift of indices on the level of filtrations. More detailed we have the following:

Lemma 2.12 *Let $\dim E = r$ and $k_\alpha \in \mathbb{Z}$ for $\alpha = 1, 2, 3$. By a shift of indices $f : \mathbb{Z}^3 \rightarrow \mathbb{Z}^3$, $(i_\alpha \rightarrow i_\alpha + k_\alpha)_{\alpha=1,2,3}$, the discriminant remains constant. In particular, we may assume that the filtrations are in standard position, i.e. for $\alpha = 1, 2, 3$ we have $E^\alpha(i) = E$ for all $i \geq 0$ and $E^\alpha(i) \neq E$ for all $i < 0$.*

Proof. Let D' be the discriminant resulting from the index shift. Since

$$\sum_i \dim E^{[\alpha]}(i) = r,$$

it follows

$$\begin{aligned} D' &= D + \sum_{\alpha} k_{\alpha}^2 r^2 + 2 \left(\sum_{\alpha} k_{\alpha} r \right) \left(\sum_{\alpha, i} i \dim E^{[\alpha]}(i) \right) \\ &\quad + 2 \sum_{\alpha \neq \beta} k_{\alpha} k_{\beta} r^2 - r \sum_{\alpha, i} 2 k_{\alpha} i \dim E^{[\alpha]}(i) - \sum_{\alpha} r^2 k_{\alpha}^2 \\ &\quad - 2r \sum_{\alpha \neq \beta, i, j} k_{\alpha} j \dim E^{[\alpha, \beta]}(i, j) + k_{\beta} i \dim E^{[\alpha, \beta]}(i, j) \\ &\quad + k_{\alpha} k_{\beta} \dim E^{[\alpha, \beta]}(i, j) \\ &= D + 2r \sum_{\alpha} k_{\alpha} \sum_{\beta \neq \alpha, i} i \dim E^{[\beta]}(i) + 2r^2 \sum_{\alpha \neq \beta} k_{\alpha} k_{\beta} \\ &\quad - 2r \sum_{\alpha \neq \beta, i, j} k_{\alpha} j \dim E^{[\alpha, \beta]}(i, j) + k_{\beta} i \dim E^{[\alpha, \beta]}(i, j) \\ &\quad + k_{\alpha} k_{\beta} \dim E^{[\alpha, \beta]}(i, j). \end{aligned}$$

Now choose n big enough such that $E^\alpha(n) = E$ and $E^\alpha(-n) = 0$ for all α . Let $d_{\alpha, \beta}(i, j) = \dim E^\alpha(i) \cap E^\beta(j)$. Obviously we have

$$\dim E^{[\alpha, \beta]}(i, j) = d_{\alpha, \beta}(i, j) - d_{\alpha, \beta}(i+1, j) - d_{\alpha, \beta}(i, j+1) + d_{\alpha, \beta}(i+1, j+1).$$

For $\alpha \neq \beta$ we have

$$\begin{aligned} \sum_{i, j} \dim E^{[\alpha, \beta]}(i, j) &= \sum_{i=-n}^n \sum_{j=-n}^n \dim E^{[\alpha, \beta]}(i, j) \\ &= d_{\alpha, \beta}(-n, -n) - d_{\alpha, \beta}(-n, n+1) \\ &\quad - d_{\alpha, \beta}(n+1, -n) + d_{\alpha, \beta}(n+1, n+1) \\ &= r. \end{aligned}$$

Then we have $d_{\alpha,\beta}(i, j) = 0$ if $i \gg 0$ or $j \gg 0$. Moreover, it follows

$$\sum_j j \dim E^{[\beta]}(j) = \sum_{j=-n}^n j(d_{\alpha,\beta}(-n, j) - d_{\alpha,\beta}(-n, j+1))$$

and we get analogously to the previous equation

$$\begin{aligned} \sum_{i,j} j \dim E^{[\alpha,\beta]}(i, j) &= \sum_{i=-n}^n \sum_{j=-n}^n j(d_{\alpha,\beta}(i, j) - d_{\alpha,\beta}(i+1, j) \\ &\quad - d_{\alpha,\beta}(i, j+1) + d_{\alpha,\beta}(i+1, j+1)) \\ &= \sum_{j=-n}^n j(d_{\alpha,\beta}(-n, j) - d_{\alpha,\beta}(-n, j+1) \\ &\quad - d_{\alpha,\beta}(n+1, j) + d_{\alpha,\beta}(n+1, j+1)) \\ &= \sum_{j=-n}^n j(d_{\alpha,\beta}(-n, j) - d_{\alpha,\beta}(-n, j+1)). \end{aligned}$$

□

We consider the subspace quiver with the vertex set

$$Q_0 = \{q_0\} \cup \{q_{i,j} \mid 1 \leq i \leq n, j \in \mathbb{N}^+\}$$

and arrow set

$$Q_1 = \{\alpha : q_{i,1} \rightarrow q_0 \mid 1 \leq i \leq n\} \cup \{\alpha : q_{i,j+1} \rightarrow q_{i,j} \mid 1 \leq i \leq n, j \in \mathbb{N}^+\}.$$

Let X be a representation with dimension vector $d = (d_i)_{i \in Q_0}$ such that $d_{i,j+1} \leq d_{i,j}$. Denote the linear maps corresponding to X by $X_{i,j} : \mathbb{C}^{d_{i,j+1}} \rightarrow \mathbb{C}^{d_{i,j}}$, where we assume that all maps $X_{i,j}$ are injective. It is easy to see that every such representation is isomorphic to a representation X' such that

$$X'_{i,j} = \begin{pmatrix} E_{d_{i,j}} \\ 0 \end{pmatrix}$$

for all $j \geq 1$, where $E_{d_{i,j}}$ is the $d_{i,j} \times d_{i,j}$ -identity matrix. In what follows we assume that all representation are of this type.

Thus a representation X is given by a n -tuple of matrices $(X_{i,0})_{1 \leq i \leq n}$ and a dimension vector $(d_i)_{i \in Q_0}$. Given a representation X we get a filtration as follows: let $(X_{i,0})_k$ be k -th column and define

$$E^i(j) = \langle (X_{i,0})_{n-d_{i,j}+1}, \dots, (X_{i,0})_n \rangle.$$

If two filtration are isomorphic, the corresponding two representations are obviously isomorphic as well (via the same g).

The other way around, we obtain a representation from a filtration. If two

representations X, X' are isomorphic, there exists a $g \in Gl_d = \Pi_{i \in Q_0} Gl_{d_i}$ with $g * X = X'$. If $d_{i,j} < d_{i,j-1}$, we have

$$g_{i,j-1} = \begin{pmatrix} g_{i,j} & \star \\ 0 & \star \end{pmatrix}$$

for all $j \geq 2$ where $g_{i,j} \in \mathbb{C}^{d_{i,j} \times d_{i,j}}$. In particular, there exists a matrix $g_0 \in Gl_{d_0}(E)$ and matrices $g_{i,1}$ such that

$$g_0 X_{i,1} = X'_{i,1} g_{i,1}$$

for all $1 \leq i \leq n$ where the $g_{i,1}$ are nested such that the subspaces $E^i(j)$ of the corresponding filtration are invariant under $g_{i,j}$.

Remark 2.13

- If we in general consider vector bundles on the projective space \mathbb{P}^n such that $n \geq 3$, Klyachko's theorem 2.11 just holds on an additional condition. The arms are in bijection with vectors generating the fan belonging to \mathbb{P}^n . Thereby we consider the projective space as a toric variety. In these cases we get a toric bundle from a filtration if all subfiltrations belonging to those arms, which correspond to vectors generating a cone of the fan, generate a distributive lattice. This is automatically satisfied in the case $n = 2$. If $n \geq 3$ this already means that every toric bundle of rank 2 splits.

We consider the stability condition given by the slope function

$$\mu(d) = \frac{\Theta(d)}{\dim d}$$

with $\Theta = -q_0^*$. The following holds, see [11]:

Theorem 2.14 *Let \mathcal{E} be a toric bundle on \mathbb{P}^2 given by a triple of filtration E^α . Then the following are equivalent:*

1. \mathcal{E} is stable in the sense of Mumford-Takemoto.
2. The family of subspaces $E^\alpha(i) \subset E$ is stable under the action of $Gl(E)$ in the sense of Mumford.
3. For all subspaces $0 \subset F \subset E$ we have

$$\sum_{\alpha, i > N} \frac{\dim E^\alpha(i) \cap F}{\dim F} < \sum_{\alpha, i > N} \frac{\dim E^\alpha(i)}{\dim E}.$$

Proof. It is easy to see that the introduced stability for the subspace quiver is equivalent to the third assertion. Therefore, we have the equivalence with two. Furthermore, it follows from the definition of the stability via first Chern class and rank that the first and third statements are equivalent.

□

Denote by $\mathcal{U}(r)$ the set of all subspace quivers with three arms, i.e. $n = 3$, with dimension vector d such that $d_{q_0} = r$ and $d_{q_{i,j}} \geq d_{q_{i,j+1}}$.

The preceding theorem means that the moduli spaces of stable representations and the moduli spaces of stable filtrations of fixed length are isomorphic. The former will be investigated in greater detail.

In particular, if $r \leq 3$ every point in $\mathcal{M}(D)^T$ corresponds to a stable representation (up to isomorphism) of the subspace quiver $\mathcal{U}(r)$, see 2.9 and 2.12. Moreover, we may understand fixed point components of $\mathcal{M}(D)$ as moduli spaces of the subspace quiver.

Note that for $r = 2$ every point in $\mathcal{M}(D)^T$ uniquely corresponds to a stable representation (up to isomorphism) of $\mathcal{U}(2)$.

For $r = 3$ and $D \equiv 0 \pmod{6}$, this correspondence is unique as well. Indeed, such a representation (filtration in standard position resp.) can be shifted such that we get a filtration with $c_1 = 0$.

For $r = 3$, $D \equiv 4 \pmod{6}$ and $c_1 \equiv 1 \pmod{3}$, such a representation can be shifted such that we get a filtration with $c_1 = 1$. Analogously, for $c_1 \equiv 2 \pmod{3}$ we get a corresponding filtration with $c_1 = 2$. Because of 2.9 this means that every point in $\mathcal{M}(D)^T$ corresponds to exactly two stable representation of $\mathcal{U}(3)$.

3 Systems of linear inequalities and polyhedrons

In this section a summary of required methods concerning systems of linear inequalities is given. For more details see for instance [20], which also provides the basis of this section. We will not place emphasise on finding a solution to the system of linear inequalities as efficient as possible because our applications in the next sections do not need it. Thus we just discuss how to get a solution of a given system of linear inequalities.

Definition 3.1 *Let $A \in \mathbb{R}^{m,n}$ be a $(m \times n)$ -matrix and $b \in \mathbb{R}^n$. A polyhedron $P \subseteq \mathbb{R}^n$ is the set of solutions $x \in \mathbb{R}^n$ of some system of linear inequalities $Ax \leq b$.*

In the following we denote the polyhedron coming from a matrix A and a vector b by $P(A, b)$.

A polytope is a bounded polyhedron. This means there exists some $s \in \mathbb{R}$ such that for every $x \in P(A, b)$ we have $\|x\| \leq s$.

Thus a polyhedron is determined by the solution of a finite number of inequalities. Every inequality defines some half space so that a polyhedron can be understood as the intersection of a finite number of half spaces.

Definition 3.2 Let $x_1, x_2, \dots, x_k \in \mathbb{R}^n$. The convex hull of these points is defined by

$$\text{conv}(x_1, \dots, x_k) := \left\{ \sum_{i=1}^k \mu_i x_i \mid \sum_{i=1}^k \mu_i = 1, \mu_i \geq 0 \right\}.$$

The convex cone is defined by

$$\text{cone}(x_1, \dots, x_k) := \left\{ \sum_{i=1}^k \mu_i x_i \mid \mu_i \geq 0 \right\}.$$

Definition 3.3 Let $P \subset \mathbb{R}^n$ be some subset. A point $x \in P$ is called an extreme point of P if for all $x_1, x_2 \in P$ and every $0 < \mu < 1$ such that $x = \mu x_1 + (1 - \mu)x_2$, we have $x = x_1 = x_2$.

Thus x can be uniquely written as a convex combination of elements of P , namely as the trivial one.

The following theorem plays an important role if we want to determine the solutions of a given system of linear inequalities, for a proof see [20].

Theorem 3.4 Let $A \in \mathbb{R}^{m,n}$ be a $(m \times n)$ -matrix and $b \in \mathbb{R}^m$ such that $m \geq n$. A point $x_0 \in P(A, b)$ is an extreme point of the polyhedron if $Ax_0 \leq b$ and $A'x_0 = b'$ for some $(n \times n)$ -submatrix of A with $\text{rank}(A') = n$ and the corresponding subvector b' of b .

By corresponding subvector we mean of course that b' results from b as follows: we remove the entry b_i if and only if we remove the i -th row of A .

In addition to the extreme points, whose convex hull corresponds to a polytope satisfying the inequalities, we determine some vectors whose positive linear combinations based on this polytope describe all solutions of the given inequalities.

Thus let $x_0, x \in \mathbb{R}^n$. Consider $x_\lambda = x_0 + \lambda x$ for $\lambda \geq 0$. Then we have $x_\lambda \in P(A, b)$ for every $\lambda \geq 0$ if and only if $x_0 \in P(A, b)$ and $Ax \leq 0$.

This leads us to the following definitions:

Definition 3.5 1. A set $C \subseteq \mathbb{R}^n$ is a cone if for every pair of points $x_1, x_2 \in C$ we have $\lambda_1 x_1 + \lambda_2 x_2 \in C$ for all $\lambda_1, \lambda_2 \geq 0$.

2. A cone is called pointed if it does not contain any subspace except $\{0\}$.

3. A half-line $y = \{\lambda x \mid \lambda \geq 0, x \in \mathbb{R}^n\}$ is an extremal ray of C if $y \in C$ and $-y \notin C$ and if for all $y_1, y_2 \in C$ and $0 < \mu < 1$ with $y = (1 - \mu)y_1 + \mu y_2$ we already have $y = y_1 = y_2$.

The polyhedral cone corresponding to some system of inequalities (A, b) is defined as

$$C(A) = \{x \in \mathbb{R}^n \mid Ax \leq 0\}.$$

Obviously $C(A)$ is both a polyhedron and a cone.

The following theorem describes how to determine all extremal rays of some polyhedral cone. For a proof again see [20].

Theorem 3.6 *Let $C(A)$ be a pointed cone. Then $x \in C(A)$ is an extremal ray of $C(A)$ if and only if there exist $\text{rank}(A) - 1$ linear independent row vectors $a_1, \dots, a_{\text{rank}(A)-1}$ of A such that*

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{\text{rank}(A)-1} \end{pmatrix} \cdot x = 0$$

and moreover $Ax \leq 0$ holds.

In what follows we assume that every polyhedron does not contain any one-dimensional subspace. Therefore, we only consider systems of inequalities whose solutions $x \in \mathbb{R}^n$ satisfy the additional condition $x_i \geq 0$. Such a system is said to be in standard form.

This assumption is no restriction because every system can be transformed into a system in standard form. The advantage of such a system is that the set of solutions does not contain lines, i.e. the corresponding polyhedron and in particular the corresponding polyhedral cone are pointed.

Moreover, note that the sets of extreme points and extremal rays are finite, which is clear because of the preceding theorems.

In conclusion we have the following:

Theorem 3.7 *Let (A, b) be a system of inequalities in standard form. Let $X = \{x_1, x_2, \dots, x_s\}$ be the set of all extreme points of the polyhedron $P(A, b)$ and $Y = \{y_1, y_2, \dots, y_t\}$ the set of all extremal rays of the polyhedral cone $C(A)$. Then the polyhedron $P(A, b)$ consisting of all solutions of the system of linear inequalities defined by $Ax \leq b$ is given by*

$$P(A, b) = \text{conv}(X) + \text{cone}(Y).$$

4 Euler characteristic of moduli spaces of stable bundles

4.1 The case of rank two bundles on the projective plane

In this section we first review the methods presented in [11] in order to deduce from it a similar formula for the Euler characteristic of rank three

bundles on the projective plane. In order to compare both results we make a small modification of Klyachko's methods.

Denote by $H(D)$ the Hurwitz function counting the number of classes of reduced binary quadratic forms Q with discriminant D with weight $\frac{2}{|\text{Aut}Q|}$. Referring to [11] the Euler characteristic of moduli spaces of stable bundles of rank two on the projective plane is given by

$$\chi(\mathcal{M}(c_1, c_2)) = \begin{cases} 3H(D), & \text{if } D \equiv -1 \pmod{4} \\ 3H(D) - \frac{3}{2}d(\frac{D}{4}), & \text{if } D \equiv 0 \pmod{4} \end{cases}.$$

The starting point for the derivation of this formula is the following well-known theorem, see for instance [3], [4] or [5]:

Theorem 4.1 *Let X be a complex variety on which a torus T acts. For the Euler characteristic χ of X we have*

$$\chi(X) = \chi(X^T).$$

Denote by E^k with $k = 1, 2, 3$ a triple of filtrations of some two-dimensional vector space E , i.e. E^k consists of E and filtrations $E^k(i)$ with $k = 1, 2, 3$ and $i \in \mathbb{Z}$ with the additional condition

$$E^k(i) = 0 \text{ for } i \gg 0 \text{ and } E^k(i) = E \text{ for } i \ll 0.$$

Fixing such a triple, we define

$$\alpha_k := |\{i \mid \dim E^k(i) = 1\}|$$

for $k = 1, 2, 3$. The stability condition corresponds to the inequalities

$$\alpha_1 < \alpha_2 + \alpha_3 \text{ and } \alpha_2 < \alpha_1 + \alpha_3 \text{ and } \alpha_3 < \alpha_1 + \alpha_2. \quad (1)$$

Therefore, for every triple $(\alpha_1, \alpha_2, \alpha_3)$ satisfying these inequalities, there exists at least one stable bundle. Note that we always assume that the filtrations are in standard form.

Hence the discriminant $-D = c_1^2 - 4c_2$ is given by

$$-D = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 2\alpha_1\alpha_2 - 2\alpha_2\alpha_3 - 2\alpha_1\alpha_3.$$

As proved in the last section, every filtration corresponds to a representation of a subspace quiver which is the quiver with three arms meeting in one point in this case. Since there exists a stable representation, by use of the dimension formula 2.4 we get that the moduli spaces of such quivers are zero-dimensional. Therefore, the Euler characteristic is one in each case. Thus we have

$$\chi(\mathcal{M}(c_1, c_2)) = |\mathcal{M}(c_1, c_2)^T|.$$

By considering the inequalities (1) and applying the theorems of the first section, we get that $P = (1, 1, 1)$ is the only extreme point of the system of linear inequalities (1).

Moreover, we obtain the extremal rays $v_1 = (1, 1, 0)$, $v_2 = (0, 1, 1)$ and $v_3 = (1, 0, 1)$. Therefore, we get that all positive integer valued solutions are of the form

$$\begin{aligned} v &= (1, 1, 1) + k_1(1, 1, 0) + k_2(0, 1, 1) + k_3(1, 0, 1) \\ &= (k_1 + k_3 + 1, k_1 + k_2 + 1, k_2 + k_3 + 1) \end{aligned}$$

with $k_1, k_2, k_3 \in \mathbb{Q}^+$.

Define

$$\mathbb{L} = \{v \in \mathbb{N}^3 \mid v = (k_1 + k_3 + 1, k_1 + k_2 + 1, k_2 + k_3 + 1), k_1, k_2, k_3 \in \mathbb{Q}^+\}.$$

Let $k_1, k_2, k_3 < 1$ and $k_i \neq 0$ for at least one k_i . Obviously the only solution we obtain in this way is $(2, 2, 2)$. Indeed, that is the case if $k_i = \frac{1}{2}$ for all $i = 1, 2, 3$. From this we get

$$\mathbb{L} = \{v \in \mathbb{N}^3 \mid v = (k_1 + k_3 + i, k_1 + k_2 + i, k_2 + k_3 + i), k_1, k_2, k_3 \in \mathbb{N}, i = 1, 2\}.$$

Indeed, if $v \in \mathbb{L}$ is a solution with $k_i \in \mathbb{Q}^+$ for $i = 1, 2, 3$, there also exists a solution $v' \in \mathbb{L}$ with $v' = (k'_1, k'_2, k'_3)$, where $k'_i := k_i - \lfloor k_i \rfloor$. Following the consideration from above we have $v' = (1, 1, 1)$ or $v' = (2, 2, 2)$. Now consider

$$\mathbb{L}_i = \{v \in \mathbb{N}^3 \mid v = (k_1 + k_3 + i, k_1 + k_2 + i, k_2 + k_3 + i), k_1, k_2, k_3 \in \mathbb{N}\}$$

with $i = 1, 2$. It is easy to see that $\mathbb{L}_1 \cap \mathbb{L}_2 = \emptyset$. If $v \in \mathbb{L}_1$, for the discriminant we get

$$-D = -4k_1k_2 - 4k_1k_3 - 4k_2k_3 - 4k_1 - 4k_2 - 4k_3 - 3$$

and for $v \in \mathbb{L}_2$ we have

$$-D = -4k_1k_2 - 4k_1k_3 - 4k_2k_3 - 8k_1 - 8k_2 - 8k_3 - 12.$$

In particular, this means that every solution $v \in \mathbb{L}_1$ belongs to a moduli space with $D \equiv 3 \pmod{4}$ and every solution $v \in \mathbb{L}_2$ to a moduli space with $D \equiv 0 \pmod{4}$.

Obviously all solutions are uniquely determined by k_1, k_2 and k_3 . Note that the second equation is equivalent to the diophantine equation

$$xy + yz + zx = n$$

for $x, y, z \geq 1$ and $n \in \mathbb{N}$ with $n \geq 3$. We can see this by dividing by -4 and defining $x = k_1 - 1, y = k_2 - 1$ and $z = k_3 - 1$ afterwards. For more details

concerning this diophantine equation see [21].

In this case we get for the generating function of the Euler characteristic

$$\begin{aligned} F(x) &= \sum_{i=0}^{\infty} \chi(\mathcal{M}(-4i)) x^{4i} \\ &= \sum_{(k_1, k_2, k_3) \in \mathbb{N}_0^3} x^{4k_1 k_2 + 4k_2 k_3 + 4k_1 k_3 + 8(k_1 + k_2 + k_3) + 12}. \end{aligned}$$

In the other case we get

$$\begin{aligned} F(x) &= \sum_{i=0}^{\infty} \chi(\mathcal{M}(-4i - 3)) x^{4i+3} \\ &= \sum_{(k_1, k_2, k_3) \in \mathbb{N}_0^3} x^{4k_1 k_2 + 4k_2 k_3 + 4k_1 k_3 + 4(k_1 + k_2 + k_3) + 3}. \end{aligned}$$

4.2 The case of stable rank three bundles on the projective plane

Let $\alpha_{ij} \geq 0$ with $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. We consider the subspace quiver with dimension vectors defined by

$$\dim(q_0) = 3,$$

$$\dim(q_{i,k}) = 2 \text{ for } 1 \leq i \leq 3, 1 \leq k \leq \alpha_{i2}$$

and

$$\dim(q_{i,k}) = 1 \text{ for } 1 \leq i \leq 3, \alpha_{i2} + 1 \leq k \leq \alpha_{i2} + \alpha_{i1}.$$

In the following denote this quiver by $\mathcal{U}(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \alpha_{31}, \alpha_{32})$.

In the following denote by U_{ij} the six different subspaces with $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. Obviously, we always have $U_{i1} \subset U_{i2}$. This means the first Chern class is given as follows:

$$c_1(\mathcal{E}) = \alpha_{11} + \alpha_{21} + \alpha_{31} + 2\alpha_{12} + 2\alpha_{22} + 2\alpha_{32}.$$

By considering the second Chern class the following problem appears: if we fix a quiver $\mathcal{U}(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \alpha_{31}, \alpha_{32})$, the second Chern class varies, depending on the number of two-dimensional subspaces that contain the one-dimensional subspaces.

We consider the cases $U_{i1} \not\subset U_{k2}$ for all i and $k \neq i$ and $U_{i1} \subset U_{j2}$ for $i \neq j$.

We first assume $U_{i1} \not\subset U_{k2}$ for all i and $k \neq i$. Then we have:

$$c_2(\mathcal{E}) = \sum_{i=1}^3 \alpha_{i2}^2 + \alpha_{i1}\alpha_{i2} + \sum_{1 \leq i < j \leq 3} \alpha_{i1}\alpha_{j1} + 2\alpha_{i1}\alpha_{j2} + 2\alpha_{i2}\alpha_{j1} + 3\alpha_{i2}\alpha_{j2}.$$

In the following let i, j and k be mutually different. If we choose U_{i1} and U_{i2} as subspaces in the stability condition with $1 \leq i \leq 3$, we get the six inequalities:

$$\alpha_{i1} + 2\alpha_{i2} < 2\alpha_{j1} + \alpha_{j2} + 2\alpha_{k1} + \alpha_{k2}$$

and

$$2\alpha_{i1} + \alpha_{i2} < \alpha_{j1} + 2\alpha_{j2} + \alpha_{k1} + 2\alpha_{k2}. \quad (2)$$

respectively. Considering the subspace $U_{i2} \cap U_{j2}$ for $i \neq j$ we get the following condition:

$$\alpha_{i2} + \alpha_{j2} < \alpha_{11} + \alpha_{21} + \alpha_{31} + 2\alpha_{k2}. \quad (3)$$

Choosing $U_{i1} \oplus U_{j1}$ we have

$$\alpha_{i1} + \alpha_{j1} < \alpha_{12} + \alpha_{22} + \alpha_{32} + 2\alpha_{k1}. \quad (4)$$

Obviously we do not have to consider other subspaces in order to test a representation for stability. Thus in the case $U_{k1} \not\subseteq U_{l2}$ for each $l \neq k$, the discriminant is given by

$$\begin{aligned} D &= 2c_1^2 - 6c_2 \\ &= \sum_{i=1}^3 2\alpha_{i1}^2 + 2\alpha_{i1}\alpha_{i2} + 2\alpha_{i2}^2 \\ &\quad - 2 \left(\sum_{1 \leq i < j \leq 3} \alpha_{i1}\alpha_{j1} + 2\alpha_{i1}\alpha_{j2} + 2\alpha_{i2}\alpha_{j1} + \alpha_{i2}\alpha_{j2} \right). \end{aligned}$$

Now consider the case $U_{i1} \subset U_{j2}$ for $i \neq j$. If $U_{i2} = U_{j2}$, we would get

$$\alpha_{i1} + 2\alpha_{i2} + \alpha_{j1} + 2\alpha_{j2} < 2\alpha_{k1} + \alpha_{k2}$$

contradicting the inequalities (2)-(4). If $U_{i1} = U_{j1}$, we analogously obtain the inequality

$$2\alpha_{i1} + \alpha_{i2} + 2\alpha_{j1} + \alpha_{j2} < \alpha_{k1} + 2\alpha_{k2}$$

again contradicting the above ones.

Thus it remains to consider the stability condition in the cases $U_{i1} \subset U_{j2}$. Then we get the additional inequalities

$$2\alpha_{i1} + \alpha_{j2} + \alpha_{i2} < 2\alpha_{k2} + \alpha_{k1} + \alpha_{i1}$$

and

$$2\alpha_{j2} + \alpha_{i1} + \alpha_{j1} < 2\alpha_{k1} + \alpha_{k2} + \alpha_{i2}. \quad (5)$$

Obviously, they do not conflict with the above inequalities. Actually, if we have nowhere equality, we obtain that exactly four of these twelve inequalities have to be satisfied. We will shortly come back to this point.

First we consider the second Chern class in the case $U_{k1} \subseteq U_{l2}$ for $k \neq l$. Then we obtain

$$\begin{aligned} c_2(\mathcal{E}) &= \sum_{i=1}^3 \alpha_{i2}^2 + \alpha_{i1}\alpha_{i2} \\ &\quad + \left(\sum_{1 \leq i < j \leq 3} \alpha_{i1}\alpha_{j1} + 2\alpha_{i1}\alpha_{j2} + 2\alpha_{i2}\alpha_{j1} + 3\alpha_{i2}\alpha_{j2} \right) - \alpha_{k1}\alpha_{l2}. \end{aligned}$$

Thus for the discriminant we obtain:

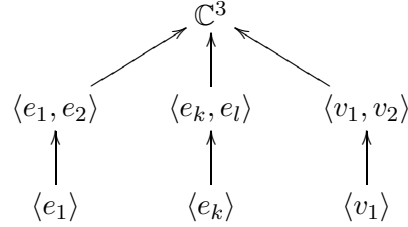
$$\begin{aligned} D &= 2c_1^2 - 6c_2 \\ &= \sum_{i=1}^3 2\alpha_{i1}^2 + 2\alpha_{i1}\alpha_{i2} + 2\alpha_{i2}^2 \\ &\quad - 2 \left(\sum_{1 \leq i < j \leq 3} \alpha_{i1}\alpha_{j1} + 2\alpha_{i1}\alpha_{j2} + 2\alpha_{i2}\alpha_{j1} + \alpha_{i2}\alpha_{j2} \right) + 6\alpha_{k1}\alpha_{l2}. \end{aligned}$$

It is easy to see that in general the discriminant satisfies the property:

$$D \equiv 0 \pmod{6} \text{ or } D \equiv 4 \pmod{6}.$$

Also note that the second Chern class does not change if $U_{k1} \subset U_{i1} \oplus U_{j1}$ for mutually different i, j and k .

Let $\alpha_{ij} = 1$ for all i, j . It is easy to see that every filtration is isomorphic to one of the following form:



Thereby $k, l \in \{1, 2, 3\}$ such that $k \neq l$ and $v_1 \neq v_2$ are arbitrary vectors. Obviously we obtain the same for filtrations of arbitrary length. Such a filtration is said to be in standard form.

4.3 The case $D \equiv 4 \pmod{6}$

Let $\mathcal{U}(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \alpha_{31}, \alpha_{32})$ be a subspace quiver with $D \equiv 4 \pmod{6}$. Considering the inequalities (2)-(5) we get the following lemma:

Lemma 4.2 *Let $\mathcal{U}(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \alpha_{31}, \alpha_{32})$ be the subspace quiver such that $D \equiv 4 \pmod{6}$. Then we have*

1. *There exist no semistable points.*

2. *There exist exactly two stable points such that $U_{i1} \subseteq U_{j2}$ for $i \neq j$.*

Proof. Let $\alpha \in \mathbb{N}^6$ such that there exists a semistable point for $\mathcal{U}(\alpha)$. We can without loss of generality assume that

$$\alpha_{11} + 2\alpha_{12} = 2\alpha_{21} + \alpha_{22} + 2\alpha_{31} + \alpha_{32}.$$

For the discriminant we get in this case by a straight forward calculation

$$\begin{aligned} D = & 6\alpha_{11}^2 + 6\alpha_{22}^2 + 6\alpha_{32}^2 - 6\alpha_{21}\alpha_{31} + 6\alpha_{22}\alpha_{32} - 6\alpha_{21}\alpha_{32} - 6\alpha_{22}\alpha_{31} \\ & - 6\alpha_{11}\alpha_{22} - 12\alpha_{11}\alpha_{32}. \end{aligned}$$

This proves the first assertion.

The second part is proved as follows: if $U_{i1} \subseteq U_{j2}$, this point cannot be semistable. Indeed, otherwise the discriminant belonging to this point would satisfy $D \equiv 0 \pmod{6}$ what is checked as before. By considering the twelve inequalities (5) in detail we see that always exactly two pairs of them have to be satisfied.

□

Let $M(\mathcal{U}(\alpha))^s$ the moduli space of stable representations of $\mathcal{U}(\alpha)$. It coincides with the moduli space of semistable representations. Following [10] and [9], therefore, it is a smooth projective variety of dimension one. By [23] we get that this projective curve (resp. the moduli space) is rational. Thus it follows, see for instance [7], that

$$M(\mathcal{U}(\alpha))^s \cong \mathbb{P}^1.$$

Denote by $D(x)$ (resp. $c_1(x)$) the discriminant (resp. first Chern-class) corresponding to a point $x \in M(\mathcal{U}(\alpha))^s$. Moreover, define

$$M(\mathcal{U}(\alpha))_D^s = \{x \in M(\mathcal{U}(\alpha))^s \mid D(x) = D\}$$

and

$$M(\mathcal{U}(\alpha))_{D,i}^s = \{x \in M(\mathcal{U}(\alpha))^s \mid D(x) = D, c_1(x) \equiv i \pmod{3}\}.$$

By the preceding section and the preceding lemma, it follows that there exists exactly one tuple $(D_1, i) \in \mathbb{N} \times \{1, 2\}$ such that $M(\mathcal{U}(\alpha))_{D_1,i}^s = \mathbb{P}^1 \setminus \{(1 : 0), (0 : 1)\}$. Moreover, there exist exactly two tuples $(D_2, j), (D_3, k) \in \mathbb{N} \times \{1, 2\}$ such that $M(\mathcal{U}(\alpha))_{D_i}^s = \{\text{pt}\}$ for $i = 2, 3$. Then by the methods of the second chapter we obtain

$$M(D)^T \cong \bigcup_{\alpha \in \mathbb{N}^6} M(\mathcal{U}(\alpha))_{D,1}^s \cong \bigcup_{\alpha \in \mathbb{N}^6} M(\mathcal{U}(\alpha))_{D,2}^s.$$

Therefore, we get

$$\chi(M(D)^T) = \frac{1}{2} \sum_{\alpha \in \mathbb{N}^6} \chi(M(\mathcal{U}(\alpha))_D^s). \quad (6)$$

Since $\chi(\mathbb{P}^1 \setminus \{(1 : 0), (0 : 1)\}) = 0$, we just need to consider such moduli spaces satisfying $M(\mathcal{U}(\alpha))_{D_i}^s = \{\text{pt}\}$ with $i = 2, 3$. They correspond to the inclusions $U_{i1} \subseteq U_{j2}$.

If we consider the twelve inequalities (5) we get the following possibilities:

1. $U_{i1} \subset U_{j2}, U_{j1} \subset U_{i2}$
2. $U_{i1} \subset U_{j2}, U_{i1} \subset U_{k2}$
3. $U_{i1} \subset U_{j2}, U_{k1} \subset U_{j2}$

for mutually different i, j and k . Therefore, the aim is to find the solutions of the system of linear inequalities consisting of the inequalities (2) – (4) and the four inequalities (5) corresponding to these three cases.

Assume $\alpha_{ij} \neq 0$, the case $\alpha_{ij} = 0$ is discussed as a special case later.

Definition 4.3 Fix $\mathcal{U}(\alpha) := \mathcal{U}(\alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{12}, \alpha_{22}, \alpha_{32})$. Let

$$\sigma_{12} \cdot (\alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{12}, \alpha_{22}, \alpha_{32}) := (\alpha_{21}, \alpha_{11}, \alpha_{31}, \alpha_{22}, \alpha_{12}, \alpha_{32}).$$

Further define σ_{13}, σ_{23} analogously and in addition

$$\tau \cdot (\alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{12}, \alpha_{22}, \alpha_{32}) := (\alpha_{12}, \alpha_{22}, \alpha_{32}, \alpha_{11}, \alpha_{21}, \alpha_{31}).$$

In doing so we get a group $G = \langle \sigma_{ij}, 1 \leq i < j \leq 3, \tau \rangle$ consisting of twelve elements.

Furthermore, we directly get the following easy lemma:

Lemma 4.4 Fix α such that the moduli space of $\mathcal{U}(\alpha)$ contains a stable representation such that $U_{i1} \subset U_{j2}$. Then the moduli spaces of $\mathcal{U}(\tau \cdot \alpha)$ and $\mathcal{U}(\sigma_{ij} \cdot \alpha)$ contain a stable point such that $U_{j1} \subset U_{i2}$, the moduli space of $\mathcal{U}(\sigma_{ik} \cdot \alpha)$ a stable point such that $U_{k1} \subset U_{j2}$ and finally the moduli space of $\mathcal{U}(\sigma_{kj} \cdot \alpha)$ contains a stable representations such that $U_{i1} \subset U_{k2}$.

Thus if we consider the above mentioned three cases, it suffices to restrict to the special cases $U_{11} \subset U_{22}, U_{32}$ and $U_{11} \subset U_{22}, U_{21} \subset U_{12}$. We consider the first case.

The solutions of the systems of inequalities are determined by the methods of Section 3. In the following we denote by P_i the sets of extreme points and by E_i the set of extremal rays with $i = 1, 2$. Then we have:

$$E_1 = \{(1, 1, 1, 1, 1, 1), (1, 3, 2, 1, 1, 1), (1, 2, 3, 1, 1, 1)\}$$

and

$$S_1 = \{(1, 1, 1, 0, 0, 0), (0, 0, 0, 1, 1, 1), (0, 1, 0, 0, 0, 1), (0, 0, 1, 0, 1, 0), \\ (0, 1, 0, 1, 0, 0), (0, 0, 1, 1, 0, 0)\}.$$

In the following denote these vectors by u_1, u_2, u_3 and w_1, \dots, w_6 respectively.

Note that, in order to calculate the extreme points and extremal rays we do not require proper inequality. In particular, the second and third extreme point correspond to semistable points.

Thus every solution v of the system of inequalities is of the form

$$\begin{aligned} v &= (s_1 + s_2 + s_3 + k_1, s_1 + 3s_2 + 2s_3 + k_1 + k_3 + k_5, \\ &\quad s_1 + 2s_2 + 3s_3 + k_1 + k_4 + k_6, s_1 + s_2 + s_3 + k_2 + k_5 + k_6, \\ &\quad s_1 + s_2 + s_3 + k_2 + k_4, s_1 + s_2 + s_3 + k_2 + k_3) \\ &= (1 + k_1, 1 + 2s_2 + s_3 + k_1 + k_3 + k_5, 1 + s_2 + 2s_3 + k_1 + k_4 + k_6, \\ &\quad 1 + k_2 + k_5 + k_6, 1 + k_2 + k_4, 1 + k_2 + k_3), \end{aligned}$$

such that $0 \leq s_i \leq 1$ for $i = 1, 2, 3$ and $k_j \geq 0$ for $i \in \{1, 2, 3, 4, 5, 6\}$ because $s_1 + s_2 + s_3 = 1$. Now we are only interested in the integer-valued and also stable solutions. We have the following:

Lemma 4.5 *All stable solutions v with $k_i < 1$ are $v_1 = (1, 1, 1, 1, 1, 1)$, $v_2 = (1, 2, 2, 1, 1, 1)$ and $v_3 = (1, 2, 2, 2, 1, 1)$. Otherwise we have $v = v_k + \sum_{i=1}^6 n_i w_i$ for a $n_i \in \mathbb{N}$ and a $k \in \{1, 2, 3\}$.*

Note that $s_i < 1$ for $i \neq 1$ has to be fulfilled in order to satisfy the stability condition because for the quivers corresponding to the extreme points except v_1 there only exist semistable representations, i.e. no stable ones.

Furthermore, we have

$$(1, 2, 2, 1, 1, 1) = \frac{1}{3}((1, 1, 1, 1, 1, 1) + (1, 2, 3, 1, 1, 1) + (1, 3, 2, 1, 1, 1))$$

and

$$\begin{aligned} (1, 2, 2, 2, 1, 1) &= \frac{2}{3}(1, 1, 1, 1, 1, 1) + \frac{1}{6}((1, 2, 3, 1, 1, 1) + (1, 3, 2, 1, 1, 1)) \\ &\quad + \frac{1}{2}((0, 0, 1, 1, 0, 0) + (0, 1, 0, 1, 0, 0)). \end{aligned}$$

The considered solutions have the upper bound $(1, 5, 5, 3, 2, 2)$. Thus we just have to verify that all other solutions are linear combinations of the desired type.

Considering the second case we obtain

$$E_2 = \{(1, 1, 1, 1, 1, 1), (1, 1, \frac{5}{2}, 1, 1, 1), (2, 1, 3, 1, 1, 1), (1, 2, 3, 1, 1, 1), \\ (1, 1, 1, 1, 2, 3), (1, 1, 1, 2, 1, 3), (1, 1, 2, 1, 1, 2), (1, 1, 1, 1, 1, \frac{5}{2})\}$$

and

$$S_2 = \{(1, 1, 1, 0, 0, 0), (0, 0, 0, 1, 1, 1), (1, 0, 0, 0, 0, 1), (0, 0, 1, 1, 0, 0), \\ (0, 1, 0, 0, 0, 1), (0, 0, 1, 0, 1, 0)\}.$$

As above we get:

Lemma 4.6 *All stable solutions with $k_i < 1$ are $v_1 = (1, 1, 1, 1, 1, 1)$, $v_2 = (1, 1, 2, 1, 1, 1)$, $v_3 = (1, 1, 1, 1, 1, 2)$, otherwise we have $v = v_k + \sum_{i=1}^6 n_i w_i$ for $n_i \in \mathbb{N}$, $k \in \{1, 2, 3\}$.*

Obviously we have $(1, 1, 2, 1, 1, 1) = \frac{1}{3}(1, 1, 1, 1, 1, 1) + \frac{2}{3}(1, 1, \frac{5}{2}, 1, 1, 1)$. Again we get an upper bound, in this case $(3, 3, 5, 3, 3, 5)$. The other solutions are again given as linear combinations.

Finally, we consider the case $\alpha_{ij} = 0$ with $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. As above we can without loss of generality assume that $\alpha_{11} = 0$ that corresponds to the case $U_{11} \subset U_{22}, U_{32}$. All other inclusions can be excluded. Therefore, the extreme points and extremal rays resp. of the inequalities are given by

$$E_3 = \{(0, 1, 1, 2, 2, 1), (0, 1, 1, 3, 1, 1), (0, 1, 1, 1, 2, 2), (0, 2, 1, 1, 1, 1), \\ (0, 1, 1, 1, \frac{3}{2}, 1), (0, 1, 1, 2, 1, 2), (0, 1, 2, 1, 1, 1), (0, 1, 1, 1, 1, \frac{3}{2}), \\ (0, 1, 1, 1, 1, 1)\}$$

and

$$S_3 = \{(0, 0, 0, 1, 1, 1), (0, 1, 0, 1, 0, 0), (0, 0, 1, 1, 0, 0), (0, 0, 1, 0, 1, 0), \\ (0, 1, 0, 0, 0, 1)\}.$$

Analogously to the other cases we obtain:

Lemma 4.7 *All stable vectors with $k_i < 1$ are $v_1 = (0, 1, 1, 1, 1, 1)$, $v_2 = (0, 1, 1, 2, 1, 1)$, $v_3 = (0, 2, 2, 3, 2, 2)$ or we have $v = v_k + \sum_{i=1}^6 n_i w_i$ with $n_i \in \mathbb{N}$, $k \in \{1, 2, 3\}$.*

It remains to prove that we get all positive integer-valued solutions as a unique linear combination in this way. The following lemma deals with this:

Lemma 4.8 *In each of the three cases the following holds: every positive integer-valued solution is a unique linear combination of the form $v = v_k + \sum_{i=1}^6 k_i w_i$ with $k_i \in \mathbb{N}$.*

Proof. Depending on the starting vector v_1 we have in each of the three cases for a linear combination α that

$$\alpha_{11} + \alpha_{21} + \alpha_{31} - \alpha_{12} - \alpha_{22} - \alpha_{32} \equiv 0, 1, 2 \pmod{3}.$$

This suffices to prove the uniqueness because in addition the extremal rays are linear independent in each case. Moreover, the cases are mutually exclusive.

□

The next aim is to calculate the discriminant in each of these cases so that we get a quadratic equation, whose number of integer-valued solutions determine the Euler characteristic of the considered moduli spaces.

Again we treat the three case from above. In the first case the solutions are of the form

$$\alpha = (k_1+1, k_1+k_3+k_5+l, k_1+k_4+k_6+m, k_2+k_5+k_6+n, k_2+k_4+1, k_2+k_3+1)$$

with $k_i \in \mathbb{N}$ and $l = m = n = 2$ or $l = m = 2$ and $n = 1$.

In the second case we have the solutions

$$\alpha = (k_1+k_3+1, k_1+k_5+1, k_1+k_4+k_6+m, k_2+k_4+1, k_2+k_6+1, k_2+k_3+k_5+n)$$

with $k_i \in \mathbb{N}$ and $n = m = 1, n = 1$ and $m = 2$ or $n = 2$ and $m = 1$.

Finally, the solutions in the third case are given by

$$\alpha = (0, k_2 + k_5 + n, k_3 + k_4 + n, k_1 + k_2 + k_3 + m, k_1 + k_4 + n, k_1 + k_5 + n)$$

with $k_i \in \mathbb{N}$ and $n = m = 1$ or $n = 1, 2$ and $m = n + 1$.

First we assume that $U_{i1} \not\subset U_{j2}$. Afterwards the discriminant in the case $U_{i1} \subset U_{j2}$ for $i \neq j$ is obtained from this case.

Let $k = (k_1, k_2, k_3, k_4, k_5, k_6)$. Then we obtain in the first case

$$\begin{aligned}
D^1(k, l, m, n) &= 2(k_1 + 1)^2 + 2(k_1 + k_3 + k_5 + l)^2 + 2(k_1 + k_4 + k_6 + m)^2 \\
&\quad + 2(k_2 + k_5 + k_6 + n)^2 + 2(k_2 + k_4 + 1)^2 + 2(k_2 + k_3 + 1)^2 \\
&\quad + 2(k_1 + 1)(k_2 + k_5 + k_6 + n) + 2(k_2 + k_4 + 1) \\
&\quad (k_1 + k_3 + k_5 + l) + 2(k_1 + k_4 + k_6 + m)(k_2 + k_3 + 1) \\
&\quad - 2(k_1 + 1)(k_1 + k_3 + k_5 + l) - 2(k_1 + 1) \\
&\quad (k_1 + k_4 + k_6 + m) - 2(k_1 + k_4 + k_6 + m) \\
&\quad (k_1 + k_3 + k_5 + l) - 2(k_2 + k_5 + k_6 + n)(k_2 + k_4 + 1) \\
&\quad - 2(k_2 + k_5 + k_6 + n)(k_2 + k_3 + 1) - 2(k_2 + k_4 + 1) \\
&\quad (k_2 + k_3 + 1) - 4(k_1 + 1)(k_2 + k_4 + 1) - 4(k_1 + 1) \\
&\quad (k_2 + k_3 + 1) - 4(k_1 + k_3 + k_5 + l)(k_2 + k_5 + k_6 + n) \\
&\quad - 4(k_1 + k_3 + k_5 + l) \\
&\quad (k_2 + k_3 + 1) - 4(k_1 + k_4 + k_6 + m)(k_2 + k_4 + 1) \\
&\quad - 4(k_1 + k_4 + k_6 + m)(k_2 + k_3 + 1) \\
\\
&= 2(k_1 + 1)(-k_1 - 3k_2 - 3k_3 - 3k_4 - l - m + n - 3) \\
&\quad + 2(k_1 + k_3 + k_5 + l) \\
&\quad (-3k_2 - k_3 - k_5 - 3k_6 + l - m - 2n - 1) \\
&\quad + 2(k_1 + k_4 + k_6 + m) \\
&\quad (k_1 - 3k_2 + k_3 - k_4 - 2k_5 + m - 2n - 1) \\
&\quad + 2(k_2 + k_5 + k_6 + n)(-k_2 - k_3 - k_4 + k_5 + k_6 + n - 2) \\
&\quad + 2(k_2 + k_4 + 1)(-k_3 + k_4) + 2(k_2 + k_3 + 1)(k_2 + k_3 + 1) \\
\\
&= -18k_1k_2 - 6k_1k_3 - 6k_1k_4 - 6k_1k_5 - 6k_1k_6 - 6k_2k_3 \\
&\quad - 6k_2k_4 - 6k_2k_5 - 6k_2k_6 - 6k_3k_5 - 6k_3k_6 - 6k_4k_5 \\
&\quad - 6k_4k_6 - 6k_5k_6 + 2k_1(-3n - 6) + 2k_2(-3l - 3m - 3) \\
&\quad + 2k_3(-3n - 3) + 2k_4(-3n - 3) + 2k_5(-3m - 3) \\
&\quad + 2k_6(-3l - 3) \\
&\quad + 2(l^2 + m^2 + n^2 - lm - 2ln - 2mn - 2l - 2m - n - 2).
\end{aligned}$$

In the second one we get

$$\begin{aligned}
D^2(k, m, n) &= 2(k_1 + k_3 + 1)^2 + 2(k_1 + k_5)^2 + 2(k_1 + k_4 + k_6 + m)^2 \\
&\quad + 2(k_2 + k_4 + 1)^2 + 2(k_2 + k_6 + 1)^2 + 2(k_2 + k_3 + k_5 + n)^2 \\
&\quad + 2(k_1 + k_3 + 1)(k_2 + k_4 + 1) + 2(k_1 + k_5 + 1)(k_2 + k_6 + 1) \\
&\quad + 2(k_1 + k_4 + k_6 + m)(k_2 + k_3 + k_5 + n) - 2(k_1 + k_3 + 1) \\
&\quad (k_1 + k_5 + 1) - 2(k_1 + k_3 + 1)(k_1 + k_4 + k_6 + m) \\
&\quad - 4(k_1 + k_3 + 1)(k_2 + k_6 + 1) - 4(k_1 + k_3 + 1) \\
&\quad (k_2 + k_3 + k_5 + n) - 2(k_1 + k_5 + 1) \\
&\quad (k_1 + k_4 + k_6 + m) - 4(k_1 + k_5 + 1)(k_2 + k_4 + 1) \\
&\quad - 4(k_1 + k_5 + 1)(k_2 + k_3 + k_5 + n) - 4(k_1 + k_4 + k_6 + m) \\
&\quad (k_2 + k_4 + 1) - 4(k_1 + k_4 + k_6 + m)(k_2 + k_6 + 1) \\
&\quad - 2(k_2 + k_4 + 1)(k_2 + k_6 + 1)2(k_2 + k_4 + 1) \\
&\quad (k_2 + k_3 + k_5 + n) - 2(k_2 + k_6 + 1)(k_2 + k_3 + k_5 + n) \\
&= 2(k_1 + k_3 + 1)(-3k_2 - k_3 - 3k_5 - 3k_6 - 1 - m - 2n) \\
&\quad + 2(k_1 + k_5 + 1)(-k_1 - 3k_2 - 2k_3 - 2k_4 - k_5 - m - 2n) \\
&\quad + 2(k_1 + k_4 + k_6 + m)(k_1 - 3k_2 - k_3 - k_4 - k_5 - n - 2) \\
&\quad 2(k_2 + k_4 + 1)(-k_2 - k_3 + k_4 - k_5 - k_6 - n) \\
&\quad 2(k_2 + k_6 + 1)(-k_3 - k_5 - n + 1) + 2(k_2 + k_3 + k_5 + n)^2 \\
&= -18k_1k_2 - 6k_1k_3 - 6k_1k_4 - 6k_1k_5 - 6k_1k_6 \\
&\quad - 6k_2k_3 - 6k_2k_4 - 6k_2k_5 - 6k_2k_6 - 6k_3k_5 - 6k_3k_6 \\
&\quad - 6k_4k_5 - 6k_4k_6 - 12k_1 - 12k_2 - 12k_3 - 12k_4 - 12k_5 \\
&\quad - 12k_6 - 6k_1n - 6k_2m - 12m - 12n + 2mn + 2n^2 + 2m^2.
\end{aligned}$$

And finally we get

$$\begin{aligned}
D^3(k, m, n) &= 2(k_2 + k_5 + n)^2 + 2(k_3 + k_4 + n)^2 + 2(k_1 + k_2 + k_3 + m)^2 \\
&\quad + 2(k_1 + k_4 + n)^2 + 2(k_1 + k_5 + n)^2 + 2(k_2 + k_5 + n) \\
&\quad (k_1 + k_4 + n) + 2(k_3 + k_4 + n)(k_1 + k_5 + n) \\
&\quad - 2(k_2 + k_5 + n)(k_3 + k_4 + n) - 2(k_1 + k_2 + k_3 + m) \\
&\quad (k_1 + k_4 + n) - 2(k_1 + k_2 + k_3 + m)(k_1 + k_5 + n) \\
&\quad - 2(k_1 + k_4 + n)(k_1 + k_5 + n) - 4(k_2 + k_5 + n) \\
&\quad (k_1 + k_2 + k_3 + m) - 4(k_2 + k_5 + n)(k_1 + k_5 + n) \\
&\quad - 4(k_3 + k_4 + n)(k_1 + k_2 + k_3 + m) \\
&\quad - 4(k_3 + k_4 + n)(k_1 + k_4 + n)
\end{aligned}$$

$$\begin{aligned}
&= 2(k_2 + k_5 + n)(-3k_1 - k_2 - 3k_3 - k_5 - 2m - n) \\
&\quad + 2(k_3 + k_4 + n)(-3k_1 - 2k_2 - k_3 - k_4 + k_5 - 2m) \\
&\quad + 2(k_1 + k_2 + k_3 + m)(-k_1 + k_2 + k_3 - k_4 - k_5 + m - 2n) \\
&\quad + 2(k_1 + k_4 + n)(k_4 - k_5) + 2(k_1 + k_5 + n)(k_1 + k_5 + n) \\
&= -6k_1k_2 - 6k_1k_3 - 6k_1k_4 - 6k_1k_5 - 6k_2k_3 \\
&\quad - 6k_2k_4 - 6k_2k_5 - 6k_3k_4 - 6k_3k_5 - 12k_1n - 12k_2n \\
&\quad - 12k_3n - 6k_4m - 6k_5m - 12mn + 2m^2.
\end{aligned}$$

Every solution of these quadratic equations with a fixed determinant D automatically satisfies the inequalities. Therefore, we "just" have to determine all solutions of these equations in order to calculate the Euler characteristic of the moduli spaces of bundles of rank three. But as mentioned above we first have to investigate the change of the discriminant in the cases in which the subspaces are embedded in each other.

Obviously in all of the cases the residue class of the discriminant only depends on l, m and n . In the first two cases we have $D \equiv 0 \pmod{6}$ if the starting vector is $(1, 1, 1, 1, 1, 1)$ and $D \equiv 4 \pmod{6}$ otherwise. In the last case we have $D \equiv 4 \pmod{6}$ if the starting vector is $(0, 1, 1, 2, 1, 1)$ or $(0, 1, 1, 1, 1, 1)$ and $D \equiv 0 \pmod{6}$ in the remaining case.

Again in the first case we have

$$\begin{aligned}
D_{11,22}^1(l, m, n, k) &= D^1(k, l, m, n) + 6\alpha_{11}\alpha_{22} \\
&= D^1(k, l, m, n) + 6(k_1k_2 + k_1k_4 + k_1 + k_2 + k_4 + 1) \\
&= -6 \sum_{1 \leq i < j \leq 6} k_i k_j + 6k_1k_4 + 6k_3k_4 - 6k_1k_2 \\
&\quad - 6n(k_1 + k_3 + k_4) - 6l(k_2 + k_6) - 6m(k_2 + k_5) \\
&\quad - 6(k_1 + k_3 + k_5 + k_6) + 2(l^2 + m^2 \\
&\quad + n^2 - lm - 2ln - 2mn - 2l - 2m - n + 1)
\end{aligned}$$

and

$$\begin{aligned}
D_{11,32}^1(l, m, n, k) &= D^1(k, l, m, n) + 6\alpha_{11}\alpha_{32} \\
&= D^1(k, l, m, n) + 6(k_1k_2 + k_1k_3 + k_1 + k_2 + k_3 + 1) \\
&= -6 \sum_{1 \leq i < j \leq 6} k_i k_j + 6k_1k_3 + 6k_3k_4 - 6k_1k_2 \\
&\quad - 6n(k_1 + k_3 + k_4) - 6l(k_2 + k_6) - 6m(k_2 + k_5) \\
&\quad - 6(k_1 + k_4 + k_5 + k_6) \\
&\quad + 2(l^2 + m^2 + n^2 - lm - 2ln - 2mn - 2l \\
&\quad - 2m - n + 1).
\end{aligned}$$

In the second case we have

$$\begin{aligned}
D_{11,22}^2(m, n, k) &= D^2(k, m, n) + 6\alpha_{11}\alpha_{22} \\
&= -6 \sum_{1 \leq i < j \leq 6} k_i k_j - 6k_1 k_2 + 6k_1 k_6 + 6k_2 k_3 + 6k_3 k_4 \\
&\quad + 6k_3 k_6 + 6k_5 k_6 - 6(k_1 + k_2 + k_3 + 2k_4 + 2k_5 + k_6) \\
&\quad - 6k_1 n - 6k_2 m - 12m - 12n + 2mn + 2n^2 + 2m^2 + 6
\end{aligned}$$

and

$$\begin{aligned}
D_{21,12}^2(m, n, k) &= D^2(k, m, n) + 6\alpha_{21}\alpha_{12} \\
&= -6 \sum_{1 \leq i < j \leq 6} k_i k_j - 6k_1 k_2 + 6k_1 k_4 + 6k_2 k_5 + 6k_3 k_4 \\
&\quad + 6k_4 k_5 + 6k_5 k_6 - 6(k_1 + k_2 + 2k_3 + k_4 + k_5 + 2k_6) \\
&\quad - 6k_1 n - 6k_2 m - 12m - 12n + 2mn + 2n^2 + 2m^2 + 6.
\end{aligned}$$

In the third case the discriminant stays constant because $\alpha_{11} = 0$.

We only have to consider the last four quadratic equations and the solutions in the third case, i.e. $\alpha_{11} = 0$, because the moduli spaces are \mathbb{P}^1 without two points. In particular, the Euler characteristic is zero.

If we evaluate the above functions at the relevant points for m, n and l , we obtain:

$$\begin{aligned}
D_{11,22}^1(2, 2, 1, k) &= -6 \sum_{1 \leq i < j \leq 6} k_i k_j + 6k_1 k_4 + 6k_3 k_4 - 6k_1 k_2 \\
&\quad - 6(2k_1 + 4k_2 + 2k_3 + k_4 + 3k_5 + 3k_6) - 22
\end{aligned}$$

and

$$\begin{aligned}
D_{11,22}^1(2, 2, 2, k) &= -6 \sum_{1 \leq i < j \leq 6} k_i k_j + 6k_1 k_4 + 6k_3 k_4 - 6k_1 k_2 \\
&\quad - 6(3k_1 + 4k_2 + 3k_3 + 2k_4 + 3k_5 + 3k_6) - 34
\end{aligned}$$

respectively, and

$$\begin{aligned}
D_{11,32}^1(2, 2, 1, k) &= -6 \sum_{1 \leq i < j \leq 6} k_i k_j + 6k_1 k_3 + 6k_3 k_4 - 6k_1 k_2 \\
&\quad - 6(2k_1 + 4k_2 + k_3 + 2k_4 + 3k_5 + 3k_6) - 22
\end{aligned}$$

and

$$\begin{aligned}
D_{11,32}^1(2, 2, 2, k) &= -6 \sum_{1 \leq i < j \leq 6} k_i k_j + 6k_1 k_3 + 6k_3 k_4 - 6k_1 k_2 \\
&\quad - 6(3k_1 + 4k_2 + 2k_3 + 3k_4 + 3k_5 + 3k_6) - 34.
\end{aligned}$$

respectively. Evaluating D^2 , we get

$$\begin{aligned} D_{11,22}^2(1, 2, k) &= -6 \sum_{1 \leq i < j \leq 6} k_i k_j - 6k_1 k_2 + 6k_1 k_6 + 6k_2 k_3 + 6k_3 k_4 \\ &\quad + 6k_3 k_6 + 6k_5 k_6 - 6(3k_1 + 2k_2 + k_3 + 2k_4 + 2k_5 + k_6) \\ &\quad - 16 \end{aligned}$$

and

$$\begin{aligned} D_{11,22}^2(2, 1, k) &= -6 \sum_{1 \leq i < j \leq 6} k_i k_j - 6k_1 k_2 + 6k_1 k_6 + 6k_2 k_3 + 6k_3 k_4 \\ &\quad + 6k_3 k_6 + 6k_5 k_6 - 6(2k_1 + 3k_2 + k_3 + 2k_4 + 2k_5 + k_6) \\ &\quad - 16 \end{aligned}$$

respectively, and

$$\begin{aligned} D_{21,12}^2(1, 2, k) &= -6 \sum_{1 \leq i < j \leq 6} k_i k_j - 6k_1 k_2 + 6k_1 k_4 + 6k_2 k_5 + 6k_3 k_4 \\ &\quad + 6k_4 k_5 + 6k_5 k_6 - 6(3k_1 + 2k_2 + 2k_3 + k_4 + k_5 + 2k_6) \\ &\quad - 16 \end{aligned}$$

and

$$\begin{aligned} D_{21,12}^2(2, 1, k) &= -6 \sum_{1 \leq i < j \leq 6} k_i k_j - 6k_1 k_2 + 6k_1 k_4 + 6k_2 k_5 + 6k_3 k_4 \\ &\quad + 6k_4 k_5 + 6k_5 k_6 - 6(2k_1 + 3k_2 + 2k_3 + k_4 + k_5 + 2k_6) \\ &\quad - 16 \end{aligned}$$

respectively. Further if $k = (k_1, k_2, k_3, k_4, k_5)$ we get

$$D^3(2, 1, k) = -6 \sum_{1 \leq i < j \leq 5} k_i k_j + 6k_4 k_5 - 12(k_1 + k_2 + k_3 + k_4 + k_5) - 16$$

and

$$D^3(1, 1, k) = -6 \sum_{1 \leq i < j \leq 5} k_i k_j + 6k_4 k_5 - 6(2k_1 + 2k_2 + 2k_3 + k_4 + k_5) - 10$$

respectively. Obviously we have the following equations:

$$D_{11,22}^1(k_1, k_2, k_3, k_4, k_5, k_6) = D_{11,32}^1(k_1, k_2, k_4, k_3, k_5, k_6),$$

$$D_{11,22}^2(k_1, k_2, k_3, k_4, k_5, k_6) = D_{21,12}^2(k_1, k_2, k_5, k_6, k_3, k_4)$$

and

$$D_{11,22}^2(1, 2, k_1, k_2, k_3, k_4, k_5, k_6) = D_{11,22}^2(2, 1, k_2, k_1, k_6, k_5, k_4, k_3).$$

Define

$$K_{i,j}^1(D, l, m, n) = \{k = (k_1, k_2, k_3, k_4, k_5, k_6) \in \mathbb{N}_0^6 \mid D_{i,j}^1(l, m, n, k) = D\}$$

for suitable i, j . Define $K_{i,j}^2(D, m, n)$ and $K^3(D, m, n)$ analogously.

By use of 6, the preceding calculations and the mentioned equalities we get:

Theorem 4.9 *Let $D \equiv 4 \pmod{6}$. Then we have*

$$\begin{aligned} \chi(\mathcal{M}(D)) &= 6|K_{11,22}^1(D, 2, 2, 1)| + 6|K_{11,22}^1(D, 2, 2, 2)| \\ &\quad + 6|K_{11,22}^2(D, 1, 2)| + 3|K^3(D, 2, 1)| + 3|K^3(D, 1, 1)|. \end{aligned}$$

Analogously to the case of rank two bundles, we obtain the following corollary concerning the generating function of the Euler characteristic:

Corollary 4.10 *Let $D \equiv 4 \pmod{6}$. We have*

$$\begin{aligned} F(x) &= 6 \left(\sum_{k \in \mathbb{N}_0^6} x^{D_{11,22}^1(2,2,1,k)} + x^{D_{11,22}^1(2,2,2,k)} + x^{D_{11,22}^2(1,2,k)} \right) \\ &\quad + 3 \sum_{k \in \mathbb{N}_0^5} x^{D^3(2,1,k)} + 3 \sum_{k \in \mathbb{N}_0^5} x^{D^3(1,1,k)}. \end{aligned}$$

Thus we have

$$\begin{aligned} F(x) &= 0x^{-4} + 3x^{-10} + 15x^{-16} + 36x^{-22} + 69x^{-28} + 114x^{-34} + 165x^{-40} \\ &\quad + 246x^{-46} + 303x^{-52} + 432x^{-58} + 492x^{-64} + 669x^{-70} + 726x^{-76} \\ &\quad + 975x^{-82} + 999x^{-88} + 1332x^{-94} + 1338x^{-100} + 1743x^{-106} \\ &\quad + 1716x^{-112} + 2226x^{-118} + 2130x^{-124} + 2775x^{-130} + 2625x^{-136} \\ &\quad + 3354x^{-142} + 3129x^{-148} + 4041x^{-154} + 3735x^{-160} + 4752x^{-166} \\ &\quad + 4317x^{-172} + 5532x^{-178} + 5070x^{-184} + 6393x^{-190} + \mathcal{O}(x^{-202}). \end{aligned}$$

4.4 The case $D \equiv 0 \pmod{6}$

In this section we discuss the case of the discriminants satisfying $D \equiv 0 \pmod{6}$. The main difference to the preceding case is that there also exist semistable points. Thus we have to modify the methods slightly.

First let $\alpha_{ij} \neq 0$ and consider all inclusions of vector spaces pointed out in the last section. All extreme points except the point $(1, 1, 1, 1, 1, 1)$ correspond to points in the case $D \equiv 4 \pmod{6}$. Thus it remains to consider this extreme point. But we have to keep in mind that the inequalities can be satisfied with equality. Thus the inequalities do not exclude each other.

Therefore, we consider all extremal rays appearing in the last section, i.e.

$$\begin{aligned} S &= \{(1, 1, 1, 0, 0, 0), (0, 0, 0, 1, 1, 1), (1, 0, 0, 0, 1, 0), (0, 1, 0, 0, 0, 1), \\ &\quad (0, 0, 1, 1, 0, 0), (1, 0, 0, 0, 0, 1), (0, 1, 0, 1, 0, 0), (0, 0, 1, 0, 1, 0)\}. \end{aligned}$$

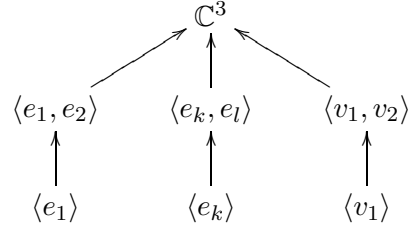
Note that the extremal rays in the nine cases of the last section arise from these eight rays by considering the first two and in addition removing one of the rays three to five and one of the rays six to eight. The linear combinations having as starting point the extreme point $(1, 1, 1, 1, 1, 1)$ thus have the following standard form

$$\begin{aligned} \alpha = & (k_1 + k_3 + k_6 + 1, k_1 + k_4 + k_7 + 1, \\ & k_1 + k_5 + k_8 + 1, k_2 + k_5 + k_7 + 1, k_2 + k_3 + k_8 + 1, \\ & k_2 + k_4 + k_6 + 1). \end{aligned}$$

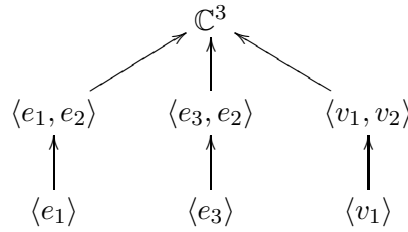
Let $k = (k_1, k_2, \dots, k_8)$. Then the discriminant is given by

$$\begin{aligned} D(k) = & 2(k_1 + k_3 + k_6 + 1)(k_1 + k_3 + k_6 + 1) + 2(k_1 + k_3 + k_6 + 1) \\ & (k_2 + k_5 + k_7 + 1) - 2(k_1 + k_3 + k_6 + 1)(k_1 + k_4 + k_7 + 1) \\ & - 2(k_1 + k_3 + k_6 + 1)(k_1 + k_5 + k_8 + 1) - 4(k_1 + k_3 + k_6 + 1) \\ & (k_2 + k_3 + k_8 + 1) - 4(k_1 + k_3 + k_6 + 1)(k_2 + k_4 + k_6 + 1) \\ & + 2(k_1 + k_4 + k_7 + 1)(k_1 + k_4 + k_7 + 1) + 2(k_1 + k_4 + k_7 + 1) \\ & (k_2 + k_3 + k_8 + 1) - 2(k_1 + k_4 + k_7 + 1)(k_1 + k_5 + k_8 + 1) \\ & - 4(k_1 + k_4 + k_7 + 1)(k_2 + k_5 + k_7 + 1) - 4(k_1 + k_4 + k_7 + 1) \\ & (k_2 + k_4 + k_6 + 1) + 2(k_1 + k_5 + k_8 + 1)(k_1 + k_5 + k_8 + 1) \\ & + 2(k_1 + k_5 + k_8 + 1)(k_2 + k_4 + k_6 + 1) - 4(k_1 + k_5 + k_8 + 1) \\ & (k_2 + k_5 + k_7 + 1) - 4(k_1 + k_5 + k_8 + 1)(k_2 + k_3 + k_8 + 1) \\ & + 2(k_2 + k_5 + k_7 + 1)(k_2 + k_5 + k_7 + 1) - (k_2 + k_5 + k_7 + 1) \\ & 2(k_2 + k_3 + k_8 + 1) - 2(k_2 + k_5 + k_7 + 1)(k_2 + k_4 + k_6 + 1) \\ & + 2(k_2 + k_3 + k_8 + 1)(k_2 + k_3 + k_8 + 1) - 2(k_2 + k_3 + k_8 + 1) \\ & (k_2 + k_4 + k_6 + 1) + 2(k_2 + k_4 + k_6 + 1)(k_2 + k_4 + k_6 + 1) \\ = & 2(k_1 + k_3 + k_6 + 1)(-k_1 - 3k_2 - k_3 - 3k_4 - k_6 - 3k_8 - 4) \\ & + 2(k_1 + k_4 + k_7 + 1)(-3k_2 + k_3 - k_4 - 3k_5 - 2k_6 - k_7 - 3) \\ & + 2(k_1 + k_5 + k_8 + 1)(k_1 - 3k_2 - 2k_3 + k_4 - k_5 + k_6 - 2k_7 \\ & - k_8 - 2) + 2(k_2 + k_5 + k_7 + 1)(-k_2 - k_3 - k_4 + k_5 - k_6 \\ & + k_7 - k_8 - 1) + 2(k_2 + k_3 + k_8 + 1)(k_3 - k_4 - k_6 + k_8) \\ & + 2(k_2 + k_4 + k_6 + 1)(k_2 + k_4 + k_6 + 1) \\ = & -6 \sum_{1 \leq i < j \leq 8} k_i k_j - 12k_1 k_2 + 6k_3 k_7 + 6k_4 k_8 + 6k_5 k_6 - 18k_1 \\ & - 18k_2 - 12 \sum_{i=3}^8 k_i - 18 \end{aligned}$$

Define $K_1 = \{k_3, k_4, k_5\}$ and $K_2 = \{k_6, k_7, k_8\}$. Again consider a filtration in standard form:



Thereby $k, l \in \{1, 2, 3\}$ such that $k \neq l$ and $v_1 \neq v_2$ are arbitrary vectors. The stable filtrations are filtrations of the form

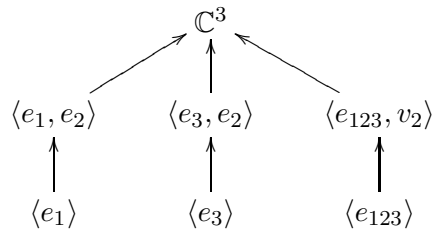


with certain conditions for v_1 and v_2 investigated in more detail now.

Define $e_{ij} = e_i + e_j$ and $e_{123} = (1, 1, 1)$. If $v_1 = e_i$, all resulting representations are unstable. If $v_1 = e_{ij}$ for $i \neq j$, the obtained filtrations correspond to polystable representations that will be analysed later. Thus let $(v_1)_i \neq 0$ and we may without loss of generality assume that $v_1 = e_{123}$.

Furthermore, we may assume that $(v_2)_1 = 0$. Indeed, we can add arbitrary multiples of e_{123} to v_2 .

Lemma 4.11 *Every stable representation of the quiver $\mathcal{U}(1, 1, 1, 1, 1, 1)$ is given by*



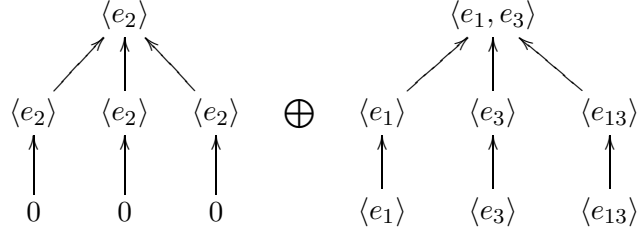
where $(v_2)_1 = 0$ and $v_2 \neq e_2, e_3, e_{23}$. In particular, we have for the moduli space of stable representations $\mathcal{M}(1, 1, 1, 1, 1, 1)^s = \mathbb{P}^1 \setminus \{(1 : 0), (0 : 1), (1 : 1)\}$.

Note that the case $v_2 = e_{23}$ is equivalent to the case $v_2 = e_1$. It is seen by an easy calculation that the three filtrations given by the points $(1 : 0), (0 : 1)$ and $(1 : 1)$ are not stable but semistable.

We obtain the following corollary:

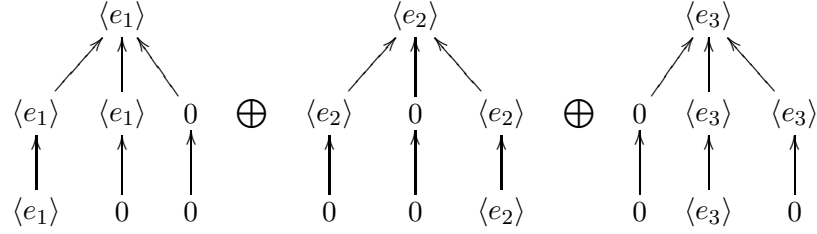
Corollary 4.12 *If there exist stable points for $\mathcal{U}(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \alpha_{31}, \alpha_{32})$, the points corresponding to the points of Lemma 4.11 are already stable.*

The polystable points can be described as follows: for $\alpha_{ij} = k$ for all i, j we obtain a polystable representation induced by the following:

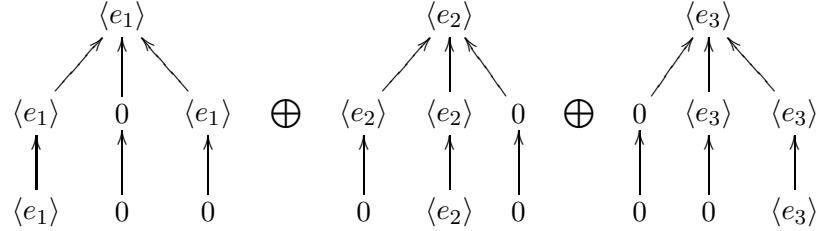


This point also induces a polystable point of the quiver obtained by extending the arms with the vector $(1, 1, 1, 1, 1, 1)$. Call these points polystable of type 1.

Further consider the polystable representations



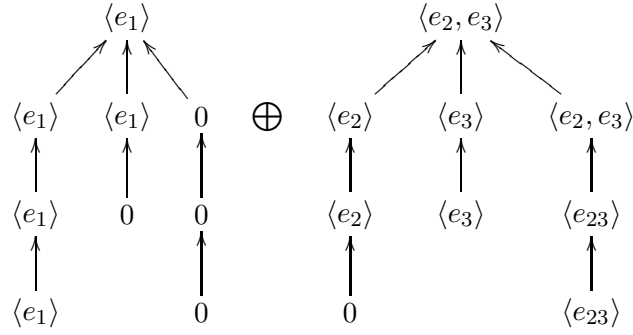
and



These are the remaining polystable points in the case $\alpha_{ij} = 1$, which persist under the extensions given by $(\alpha_{21}, \alpha_{12})$, $(\alpha_{31}, \alpha_{22})$ and $(\alpha_{11}, \alpha_{32})$ and $(\alpha_{11}, \alpha_{22})$, $(\alpha_{21}, \alpha_{32})$ and $(\alpha_{31}, \alpha_{12})$ respectively. Call these points polystable of type 2.

If we consider the case with lengths of arms given by $(1, 1, 2, 2, 1, 1)$, we get

the polystable point



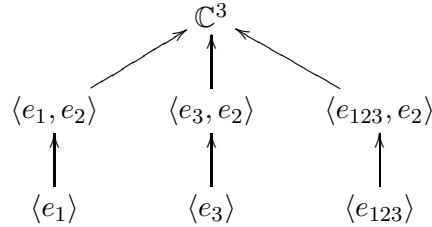
and the corresponding polystable points according to the remaining five extensions.

They persist under extensions given by $(\alpha_{11}, \alpha_{32})$, $(\alpha_{31}, \alpha_{22})$, $(\alpha_{21}, \alpha_{12})$ and $(\alpha_{31}, \alpha_{12})$. Call these points polystable of type 3. Note that polystable points of type 2 and 3 also persist under extending the arms by the vectors $(1, 1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1, 1)$ respectively.

If we want to determine a solution of the given system of inequalities, we may assume that there exists at least one $k \in K_i$ for every $i = 1, 2$ with $k = 0$. If we again consider the inequalities (2)-(5) and the investigations of the polystable points, we get in conclusion:

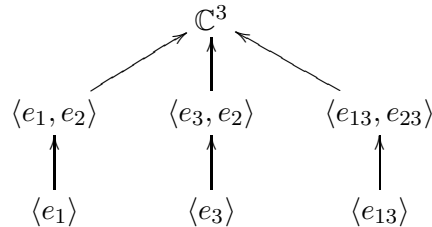
Lemma 4.13 *Let $\alpha \in \mathbb{N}_0^6$ be in standard form.*

1. *The filtration induced by*



is stable if and only if $k_1 > k_2$.

2. *The filtration induced by*



is stable if and only if $k_2 > k_1$.

3. If $k_1 = k_2$, there exists exactly one polystable point of type 1.
4. If $k \neq 0$ for exactly one $k \in K_i$, $i = 1, 2$, there exists a polystable point of type 2 and a polystable point of type 3.
5. If $k, l \neq 0$ for exactly two different $k, l \in K_i$, $i = 1, 2$, there exists a stable point such that $U_{i1} \subset U_{j2}$ with $i \neq j$ and a polystable point of type 2.
6. If $k, l \neq 0$ for exactly one $k \in K_1$ and exactly one $l \in K_2$, there exist two polystable points of type 3.
7. If $k, l, n \neq 0$ for exactly two different $k, l \in K_i$ and one $n \in K_j$ with $i \neq j$, there exists a stable point such that $U_{i1} \subset U_{j2}$ with $i \neq j$ and a polystable point of type 3.
8. If $k, l, m, n \neq 0$ for exactly two different $k, l \in K_1$ and exactly two different $n, m \in K_2$, there exist two stable points such that $U_{i1} \subset U_{j2}$ with $i \neq j$.

Considering all polystable filtrations treated in this section, an easy calculation using the results of [1] shows that all of these points are smooth. If we assume that there exists a least one stable filtration, in the same manner as in the last section we get that

$$M(\mathcal{U}(\alpha))^{ss} \cong \mathbb{P}^1.$$

The moduli space of stable points is obtained by the considerations of the last lemma.

Note that the moduli space of stable points and the one of semistable points coincide if in the eighth case of the lemma $k_1 \neq k_2$ holds.

Finally, we have to consider the cases when $\alpha_{ij} = 0$ for exactly one pair i, j . This is the case of the extreme point $(0, 2, 2, 3, 2, 2)$. As in the last section, we may assume $\alpha_{11} = 0$. Then the discriminant is given by

$$\begin{aligned} D^0(k) = & -6 \sum_{1 \leq i < j \leq 5} k_i k_j + 6k_4 k_5 - 24k_1 - 24k_2 \\ & - 24k_3 - 18k_4 - 18k_5 - 54. \end{aligned}$$

Again we determine the generating function. Therefore, define

$$D^1(k) = D(k) + 6(k_1 + k_5 + k_8 + 1)(k_2 + k_5 + k_7 + 1),$$

$$D^2(k) = D(k) + 6(k_1 + k_3 + k_6 + 1)(k_2 + k_4 + k_6 + 1)$$

and

$$D^3(k) = D(k) + 6(k_1 + k_5 + k_8 + 1)(k_2 + k_3 + k_8 + 1).$$

This corresponds to the discriminant in the case of inclusions $U_{31} \subset U_{12}$, $U_{11} \subset U_{32}$ and $U_{31} \subset U_{22}$. It suffices to consider the cases provided by this three cases, the other ones can again be constructed via permutation of the arms.

Let

$$F(x) = \sum_{n=0}^{\infty} \chi(\mathcal{M}(6n)) x^{-6n}$$

be the generating function of the Euler characteristic. Define $\mathcal{N}^{k,l} = \mathbb{N}^k \times \mathbb{N}_+^l$ for a $k \in \mathbb{N}_+$.

Then we get the following result:

Theorem 4.14 *We have*

$$\begin{aligned} F(x) = & - \sum_{k=0}^{\infty} x^{D(k,k,0,\dots,0)} - 6 \sum_{k \in \mathcal{N}^{1,1}} x^{D(k_1,k_1,k_2,0,\dots,0)} \\ & - 3 \sum_{k \in \mathcal{N}^{1,2}} x^{D(k_1,k_1,k_2,0,0,0,k_3,0)} - 6 \sum_{k \in \mathcal{N}^{1,2}} x^{D(k_1,k_1,k_2,0,0,0,0,k_3)} \\ & - 6 \sum_{k \in \mathcal{N}^{1,2}} x^{D(k_1,k_1,k_2,k_3,0,0,0,0)} - 12 \sum_{k \in \mathcal{N}^{1,3}} x^{D(k_1,k_1,k_2,k_3,0,0,k_4,0)} \\ & - 6 \sum_{k \in \mathcal{N}^{1,3}} x^{D(k_1,k_1,k_2,k_3,0,k_4,0,0)} - 3 \sum_{k \in \mathcal{N}^{1,4}} x^{D(k_1,k_1,k_2,k_3,0,0,k_4,k_5)} \\ & - 6 \sum_{k \in \mathcal{N}^{1,4}} x^{D(k_1,k_1,k_2,k_3,0,k_4,k_5,0)} \\ & + 6 \sum_{k \in \mathcal{N}^{2,2}} x^{D^1(k_1,k_2,k_3,k_4,0,0,0,0)} + 12 \sum_{k \in \mathcal{N}^{2,3}} x^{D^1(k_1,k_2,k_3,k_4,0,0,k_5,0)} \\ & + 6 \sum_{k \in \mathcal{N}^{2,3}} x^{D^1(k_1,k_2,k_3,k_4,0,k_5,0,0)} + 3 \sum_{k \in \mathcal{N}^{2,4}} x^{D^1(k_1,k_2,k_3,k_4,0,0,k_5,k_6)} \\ & + 3 \sum_{k \in \mathcal{N}^{2,4}} x^{D^2(k_1,k_2,k_3,k_4,0,0,k_5,k_6)} + 6 \sum_{k \in \mathcal{N}^{2,4}} x^{D^1(k_1,k_2,k_3,k_4,0,k_5,k_6,0)} \\ & + 6 \sum_{k \in \mathcal{N}^{2,4}} x^{D^3(k_1,k_2,k_3,k_4,0,k_5,k_6,0)} + 6 \sum_{k \in \mathbb{N}^5} x^{D^0(k)}. \end{aligned}$$

As far as the case $k_1 \neq k_2$ is concerned, note that we only have to count the stable points coming from inclusions. Indeed, the moduli space is the projective line without two points.

Thus we have

$$\begin{aligned}
F(x) = & 0x^0 + 0x^{-6} + 0x^{-12} - 1x^{-18} + 0x^{-24} - 6x^{-30} + 0x^{-36} - 3x^{-42} \\
& - 12x^{-48} + 6x^{-54} + 12x^{-60} - 15x^{-66} + 17x^{-72} + 72x^{-78} - 24x^{-84} \\
& + 102x^{-90} + 30x^{-96} + 138x^{-102} + 132x^{-108} + 171x^{-114} + 27x^{-120} \\
& + 420x^{-126} + 204x^{-132} + 360x^{-138} + 180x^{-144} + 678x^{-150} \\
& + 192x^{-156} + 773x^{-162} + 351x^{-168} + 906x^{-174} + 624x^{-180} \\
& + 816x^{-186} + 519x^{-192} + \mathcal{O}(x^{-198}).
\end{aligned}$$

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